Math 210C Lecture 15 Notes

Daniel Raban

May 3, 2019

1 Splitting of Morphisms and Projective and Injective Objects

1.1 Splitting of morphisms

Let $\mathcal{C}$ be an abelian category.

Definition 1.1. An epimorphism $\pi : B \to C$ is **split** if there exists $t : C \to B$ such that $\pi \circ t = \mathrm{id}_C$.

Definition 1.2. A monomorphism $\iota : A \to B$ is **split** if there exists $s : B \to A$ such that $s \circ \iota = \mathrm{id}_A$.

Proposition 1.1. *Let*

$$0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} C \longrightarrow 0$$

be exact. Then the following are equivalent:

1. *ι splits.*
2. *π splits.*
3. *There is an isomorphism $w : A \oplus C \to B$ such that $w \circ \iota_A = \iota : A \to B$ and $\pi \circ w \circ \iota_C = \mathrm{id}_C$, where $\iota_A : A \to A \oplus C$ and $\iota_C : C \to A \oplus C$ are the natural inclusions. Moreover, $\mathrm{id}_B = \iota \circ s + t \circ \pi$.*

We will give the proof in R-Mod.

Proof. (1) $\implies$ (2): Let s be such that $s \circ \iota = \mathrm{id}_A$. Then let $\tilde{t} : B \to B$ be $\tilde{t} = \mathrm{id}_C - \iota \circ s$. Then $\tilde{t}(\iota(a)) = \iota(a) - \iota(s(\iota(a))) = \iota(a) - \iota(a) = 0$. Then $\tilde{t}|_{\iota(A)} = 0$. So there is a $t : C \to B$ such that if $t \circ \pi = \tilde{t}$. Now if $b \in B$ is such that $\pi(b) = c$, then $\pi \circ t(c) = \pi \circ \tilde{t}(b) = \pi(\mathrm{id}_B - \iota \circ s)(b) = \pi(b) = c$.

(2) $\implies$ (1): Let t be such that $\pi \circ t = \mathrm{id}_C$. Then let $\tilde{s} : B \to B$ be $\tilde{s} = \mathrm{id}_B - t \circ \pi$. Then $\tilde{s}(\iota(a)) = \iota(a) - t \circ \pi \circ \iota(a) = \iota(a)$. So $\pi \circ \tilde{s}(b) = \pi(b) - \pi \circ t \circ \pi(b) = 0$, so $\pi \circ \tilde{s} = 0$. Then $\tilde{s}$ takes values in $\iota(A)$, so there is an $s : B \to A$ such that $\iota \circ s = \tilde{s}$.

So $\tilde{s}(t(c)) = t(c) - t \circ \pi \circ t(c) = t(c) - t(c) = 0$, which means $\tilde{s}|_{t(C)} = 0$. $s \circ \iota = \mathrm{id}_A$

(2) $\implies$ (3): We have $\pi \circ t = \mathrm{id}_C$ and $s : B \to A$ such that $s \circ \iota = \mathrm{id}_A$. Define $w(a, c) = \iota(a) + t(c)$. Then we can define the inverse $v : B \to A \oplus C$ by $v(b) = (s(b), \pi(b)$. Check that $w \circ v = \iota \circ s + t \circ \pi = \mathrm{id}_B$, ad $v \circ w(a, c) = v(\iota(a) + t(b)) = (a, c)$.

(3) $\implies$ (2): Let $t = w \circ \iota_C$. Then $\pi \circ t = \pi \circ w \circ \iota_C = \mathrm{id}_C$. $\square$

Definition 1.3. Sequences satisfying any of these properties are called **split** short exact sequences.

Example 1.1. in Ab,

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/6\mathbb{Z} \xrightarrow{\bmod 3} \mathbb{Z}/3\mathbb{Z} \longrightarrow 0$$

splits with $t = \times 4$ and $s = \mod 2$. However, the sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

is not split.

Remark 1.1. In Grp,

$$1 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} G/N \longrightarrow 1$$

is split iff $G \cong N \rtimes H$, where $H \cong G/N$.

1.2 Projective and injective objects

Definition 1.4. $P \in \mathcal{C}$ is **projective** if for every epimorphism $\pi : B \to C$ and morphism $g : P \to C$, there is an $f : P \to B$ such that $\pi \circ f = g$.

$$\begin{tikzcd} & P \arrow[dl, dashed, "f"'] \arrow[d, "g"] & \\ B \arrow[r, "\pi"] & C \arrow[r] & 0 \end{tikzcd}$$

Definition 1.5. $I \in \mathcal{C}$ is **injective** if for every monomorphism $\iota : A\ toV$ and $f : A \to I$, there exists a $g : B \to I$ such that $g \circ \iota = f$.

$$\begin{tikzcd} & & P \\ 0 \arrow[r] & A \arrow[ur, "f"] \arrow[r, "\iota"] & B \arrow[u, dashed, "g"'] \end{tikzcd}$$

Proposition 1.2. *$P \in \mathcal{C}$ is projective if and only every epimorphism $\pi : B \to P$ splits. $I \in \mathcal{C}$ is injective if and only if every monomorphism $\iota : P \to B$ splits.*

Proof. ($\Longrightarrow$): There exists g by projectivity of P such that $\pi \circ g = \mathrm{id}_P$:

$$\begin{tikzcd}
& P \arrow[dl, dashed, "f"'] \arrow[d, "\mathrm{id}_P"] & \\
B \arrow[r, "\pi"] & P \arrow[r] & 0
\end{tikzcd}$$

($\Longleftarrow$): Look at the diagram

$$\begin{tikzcd}
& X \arrow[dl, "\overline{g}"'] \arrow[dr, "\overline{\pi}"] & \\
B \arrow[dr, "\pi"'] & & P \arrow[ul, dashed, bend right, "t"'] \arrow[ll, dashed, "f"] \arrow[dl, "g"] \\
& C &
\end{tikzcd}$$

where X is the push-out of P and B over C. $\square$

Lemma 1.1. *In R-*Mod*, every free module is projective.*

Proof. Let $\pi : B \to \bigoplus_{x \in X} R \cdot x$. For each $x \in X$ let $b_x \in B$ such that $\pi(b_x) = x$. Define $t : \bigoplus R \cdot x \to B$ by $t(x) = b_x$ for all $x \in X$. $\square$

Proposition 1.3. *An R-module is projective if and only if it is a direct summand of a free module.*

Proof. ($\Longrightarrow$): Suppose P is projective. There exists a free R-module F and an epimorphism $\pi : F \to P$. Since P is projective, there exists a $t : P \to F$ such that $\pi \circ t = \mathrm{id}_P$. So $F = P \oplus Q$ for some Q.

($\Longleftarrow$): Now suppose that $F = P \oplus Q$, where F is free. If $\pi : B \to P$ is an epimorphism, then $\pi : B \oplus Q \to P \oplus Q = F$ is a surjection, so there is a splitting $t : F \to B \oplus Q$. Note that $(\pi \circ \mathrm{id}_Q)(t(p)) = p$, so $t(p) \in P$ and $\pi \circ t|_P = \mathrm{id}_P$. $\square$

Example 1.2. $\mathbb{Z}/n\mathbb{Z}$ is not $\mathbb{Z}$-projective.

Example 1.3. $\mathbb{Z}$ is $\mathbb{Z}$-projective.

Example 1.4. $\mathbb{Q}$ is not $\mathbb{Z}$-projective, as it is not a direct summand of a free module.

Example 1.5. Take $M_n(F)$, where F is a vield. Then $V = F^{\oplus n}$ be the set of column vectors. Then $M_n(F) = V^{\oplus n}$, so V is a projective $M_n(F)$-module.